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Rationalized evaluation subgroups of a map II: Quillen models and adjoint maps

Gregory Lupton, Samuel Bruce Smith

1. Introduction

In a previous paper [8], we have used the Sullivan model of a map to develop a framework within which rational homotopy groups of function spaces and related topics may be studied. Here, we construct a corresponding framework using differential graded Lie algebras. We use the notation and vocabulary established in [8] freely and without comment. Our main result, Theorem 3.1, identifies the homomorphism induced on rational homotopy groups by the fibre inclusion $\mathrm{map}_*(X, Y; f) \rightarrow \mathrm{map}(X, Y; f)$ of the evaluation fibration. The basic idea behind this theorem may be indicated as follows: a map $f: S^n \times X \rightarrow Y$ that restricts to the trivial map on S^n yields a certain morphism of degree n from the Quillen minimal model of X to that of Y ; this morphism turns out to be a derivation. The space of these derivations can be endowed with a differential, and the homology of this chain complex yields the rational homotopy groups of the based mapping space $\mathrm{map}_*(X, Y; f)$. Theorem 3.3 extends Tanré's description of the rationalized Gottlieb group [13, VII.4.(10)] to a description of the rationalized evaluation subgroup of a map.

In Section 4 we consider a generalization of a question of Gottlieb, concerning the difference between the Gottlieb group and the Whitehead center, in the context of rational homotopy theory. In Section 5 we describe and study the

rationalization of the so-called *G-sequence of a map* as constructed by Lee and Woo [14]. We obtain results about the *G-sequence* that complement those of [8]. In Section 2 we describe an explicit Quillen minimal model for a product $A \times X$, in terms of the Quillen models of the factors, in the case in which A is a rational co- H -space. The paper also includes a technical appendix in which we prove a particular result from DG Lie algebra homotopy theory necessary for our proof of Theorem 3.1.

We finish this introduction by setting some notation, especially with respect to DG Lie algebras. We remind the reader that we use the same notational conventions and running hypotheses as [8]. A vector space (over the rationals) generated by a single element v will be written $\mathbb{Q}v$. The k th *suspension* of V , denoted by $s^k(V)$, is the vector space defined as $(s^k(V))_n = V_{n-k}$. A *DG Lie algebra* is a pair (L, d) where L is a graded Lie algebra and d is a vector space differential that satisfies the derivation law $d([x, y]) = [d(x), y] + (-1)^{|x|}[x, d(y)]$. We write $\mathbb{L}(V)$ for the free Lie algebra generated by the vector space V . The coproduct (or “free product”) of (DG) Lie algebras L and L' is written $L \sqcup L'$. We write $\mathbb{L}(V, W)$ for $\mathbb{L}(V) \sqcup \mathbb{L}(W) = \mathbb{L}(V \oplus W)$ and $(\mathbb{L}(V), d)$ as $\mathbb{L}(V; d)$. A DG Lie algebra (L, d) is *minimal* if L is free and the differential is decomposable, that is, $d(L) \subseteq [L, L]$.

We assume the reader is familiar with the basic facts of rational homotopy theory from the Quillen point of view, that is, using DG Lie algebra minimal models. Good references for this material include [13] and [3, Part IV]. Specifically, we recall that each space X has a *Quillen minimal model* which is a minimal DG Lie algebra (\mathcal{L}_X, d_X) whose isomorphism type is a complete invariant of the rational homotopy type of X . As a Lie algebra, we have $\mathcal{L}_X = \mathbb{L}(s^{-1}\tilde{H}_*(X; \mathbb{Q}))$. A map of spaces $f: X \rightarrow Y$ induces a map of Quillen models which we denote $\mathcal{L}_f: \mathcal{L}_X \rightarrow \mathcal{L}_Y$.

Given a map $\psi: (L, d_L) \rightarrow (K, d_K)$ of DG Lie algebras, define a *ψ -derivation of degree n* to be a linear map $\theta: L \rightarrow K$ that increases degree by n and satisfies $\theta([\alpha, \beta]) = [\theta(\alpha), \psi(\beta)] + (-1)^{n|\alpha|}[\psi(\alpha), \theta(\beta)]$ for $\alpha, \beta \in L$. Let $\text{Der}_n(L, K; \psi)$ denote the space of all ψ -derivations of degree n from L to K . Next, define $D: \text{Der}_n(L, K; \psi) \rightarrow \text{Der}_{n-1}(L, K; \psi)$ by $D(\theta) = d_K \circ \theta - (-1)^{|\theta|}\theta \circ d_L$. The pair $(\text{Der}_*(L, K; \psi), D)$ is then a DG vector space. The *adjoint map associated to ψ* is $\text{ad}_\psi: K \rightarrow \text{Der}_*(L, K; \psi)$ where $\text{ad}_\psi(\alpha)(\beta) = [\alpha, \psi(\beta)]$. It is easy to check that ad_ψ is a map of DG vector spaces. We write ad for the adjoint map associated to the identity $1: L \rightarrow L$.

We will need the *mapping cone* of a map of DG vector spaces $\phi: V \rightarrow W$ (cf. [12, p. 166] or [8, Def. 3.2]). This is the DG vector space $(\text{Rel}_*(\phi), \delta)$ with $\text{Rel}_n(\phi) = V_{n-1} \oplus W_n$ and differential of degree -1 defined as $\delta(v, w) = (-d_V(v), \phi(v) + d_W(w))$. The inclusion $J: W_n \rightarrow \text{Rel}_n(\phi)$ and the projection $P: \text{Rel}_n(\phi) \rightarrow V_{n-1}$ with $J(w_n) = (0, w_n)$, respectively, $P(v_{n-1}, w_n) = v_{n-1}$, give a short exact sequence of chain complexes that leads to a long exact sequence on homology with connecting homomorphism $H(\phi)$.

Now apply this construction to the adjoint map $\text{ad}_\psi: K \rightarrow \text{Der}_*(L, K; \psi)$ from above. We obtain a long exact homology sequence

$$\cdots \longrightarrow H_n(K) \xrightarrow{H(\text{ad}_\psi)} H_n(\text{Der}(L, K; \psi)) \xrightarrow{H(J)} H_n(\text{Rel}(\text{ad}_\psi)) \xrightarrow{H(P)} \cdots$$

that we call the *long exact derivation homology sequence of ψ* . In Theorem 3.2 below, we show that this sequence corresponds to the long exact homotopy sequence of the evaluation fibration when $\psi: L \rightarrow K$ is the Quillen model of a map.

2. Quillen models for certain products

We describe a Quillen minimal model of $A \times X$ when A is any rational co- H -space of finite type. Our description essentially makes explicit that of [13, Prop. VII.1(2)], although our treatment here is self-contained.

Suppose X has Quillen minimal model $\mathbb{L}(W; d_X)$, and that $\mathbb{L}(V; d = 0)$ is the Quillen minimal model of a simply connected, finite-type rational co- H -space A . Suppose $\{v_i\}_{i \in J}$ is a (connected, finite-type) basis for V , and $|v_i| = n_i - 1$ for each $i \in J$. Topologically, this corresponds to A being of the rational homotopy type of the wedge of spheres $\bigvee_{i \in J} S^{n_i}$. We construct a new minimal DG Lie algebra from these data as follows: For each $i \in J$, set $W_i = s^{n_i}(W)$. Let $\lambda: \mathbb{L}(W) \rightarrow \mathbb{L}(W, V, \bigoplus_{i \in J} W_i)$ denote the inclusion of graded Lie algebras. For each $i \in J$, let $S_i: W \rightarrow W_i$ denote the n_i -fold suspension isomorphism and define a λ -derivation $S_i: \mathbb{L}(W) \rightarrow \mathbb{L}(W, V, \bigoplus_{i \in J} W_i)$ by extending S_i using the λ -derivation rule. Now define a differential ∂ on $\mathbb{L}(W, V, \bigoplus_{i \in J} W_i)$ that extends the differentials

on $\mathbb{L}(W)$ and $\mathbb{L}(V)$, by setting

$$\partial(S_i(w)) = (-1)^{n_i-1}[v_i, w] + (-1)^{n_i}S_i(d_X(w))$$

for each generator $w \in W$ (and thus each generator of W_i). Note that this may also be expressed as a boundary relation $D(S_i) = (-1)^{n_i-1}\text{ad}_\lambda(v_i)$ in $\text{Der}_{n_i-1}(\mathbb{L}(W), \mathbb{L}(W, V, \oplus_{i \in J} W_i); \lambda)$.

We will show that $\mathbb{L}(W, V, \oplus_{i \in J} W_i; \partial)$ is the Quillen minimal model of $A \times X$. First, note that ∂ is a differential, since we have

$$\begin{aligned} (\partial)^2(S_i(w)) &= \partial((-1)^{n_i-1}[v_i, w] + (-1)^{n_i}S_i(d_X(w))) \\ &= [v_i, d_X(w)] + (-1)^{n_i}\partial S_i(d_X(w)) = S_i d_X(d_X(w)) = 0. \end{aligned}$$

Theorem 2.1. *Let X be a simply connected space of finite type with Quillen minimal model $\mathbb{L}(W; d_X)$. Let A be a rational co- H -space of finite type and of the rational homotopy type of the wedge of spheres $\bigvee_{i \in J} S^{n_i}$. Then $\mathbb{L}(W, V, \oplus_{i \in J} W_i; \partial)$, as above, is the Quillen minimal model of $A \times X$.*

Proof. $\mathbb{L}(V) \oplus \mathbb{L}(W; d_X)$ is a non-minimal model for $A \times X$ [3, p. 332, Ex. 3]. We will show that the obvious projection

$$p: \mathbb{L}(W, V, \oplus_{i \in J} W_i; \partial) \rightarrow \mathbb{L}(V) \oplus \mathbb{L}(W; d_X)$$

is a quasi-isomorphism. Since the domain is a minimal DG Lie algebra, this will suffice to show that it is the Quillen minimal model of the product.

Consider the following commutative diagram of DG Lie algebra maps:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{i} & \mathbb{L}(W, V, \oplus_{i \in J} W_i; \partial) & \xrightarrow{q} & \mathbb{L}(V) & \longrightarrow & 0 \\ & & \downarrow p' & & \downarrow p & & \downarrow 1 & & \\ 0 & \longrightarrow & \mathbb{L}(W; d_X) & \xrightarrow{i'} & \mathbb{L}(V) \oplus \mathbb{L}(W; d_X) & \xrightarrow{q'} & \mathbb{L}(V) & \longrightarrow & 0 \end{array} \quad (1)$$

Here, q and q' are the obvious (quotient) projections onto $\mathbb{L}(V)$, and i and i' are the inclusions of the kernels, so that the rows are short exact sequences of DG Lie algebras. We will argue that $p': K \rightarrow \mathbb{L}(W; d_X)$ is a quasi-isomorphism. As a sub-DG Lie algebra of a connected, free DG Lie algebra, K is itself a connected, free DG Lie algebra. Indeed, we may write

$$K = \mathbb{L}(W, \oplus_i W_i, [V, W], \oplus_i [V, W_i], [V, [V, W]], \oplus_i [V, [V, W_i]], \dots; \partial_K)$$

or more succinctly $K = \mathbb{L}(\{\text{ad}^j(V)(W)\}_{j \geq 0}, \{\oplus_i \text{ad}^j(V)(W_i)\}_{j \geq 0}; \partial_K)$. We now claim that $(\partial_K)_0$, the linear part of the differential in K , induces isomorphisms

$$(\partial_K)_0: \oplus_{i \in J} \text{ad}^j(V)(W_i) \rightarrow \text{ad}^{j+1}(V)(W)$$

for each $j \geq 0$. Suppose $(v_{r_1}, v_{r_2}, \dots, v_{r_j}) \in V^j$ is a j -tuple and $w \in W$. Then write $\text{ad}(v_{r_1}, v_{r_2}, \dots, v_{r_j})(w)$ for $[v_{r_1}, [v_{r_2}, [\dots, [v_{r_{j-1}}, [v_{r_j}, w]] \dots]]$ and likewise for elements of $\text{ad}^j(V)(W_i)$. From the definition of ∂ above, we have

$$\begin{aligned} \partial(\text{ad}(v_{r_1}, v_{r_2}, \dots, v_{r_j})(S_i(w))) &= \pm \text{ad}(v_{r_1}, v_{r_2}, \dots, v_{r_j})(\partial(S_i(w))) \\ &= \pm \text{ad}(v_{r_1}, v_{r_2}, \dots, v_{r_j}, v_i)(w) \\ &\quad \pm \text{ad}(v_{r_1}, v_{r_2}, \dots, v_{r_j})(S_i d_X(w)). \end{aligned}$$

Now $d_X(w)$ is decomposable in $\mathbb{L}(W)$ and thus $S_i d_X(w)$ is decomposable in $\mathbb{L}(W, W_i) \subseteq K$. Since K is an ideal, the last term displayed above, namely $\text{ad}(v_{r_1}, v_{r_2}, \dots, v_{r_j})(S_i d_X(w))$, is decomposable in K . It follows that the linear part of the differential in K induces isomorphisms $(\partial_K)_0: \text{ad}^j(V)(W_i) \cong \text{ad}^j(V)\text{ad}(v_i)(W)$ for each $i \in J$ and each $j \geq 0$, and hence isomorphisms $(\partial_K)_0: \oplus_i \text{ad}^j(V)(W_i) \cong \text{ad}^{j+1}(V)(W)$ for each $j \geq 0$, as claimed. Notice that as a consequence of this, we must have $(\partial_K)_0 = 0$ on each vector space of generators $\text{ad}^{j+1}(V)(W)$ in K , for $j \geq 0$, since the linear part of a differential is itself a differential. Finally, notice that $(\partial_K)_0 = 0$ on the vector space of generators

W in K . It now follows that the DG vector space $(Q(K), (\partial_K)_0)$ obtained by linearizing K may be written as a direct sum

$$(Q(K), (\partial_K)_0) \cong W \oplus_{j \geq 0} \left((\oplus_i \operatorname{ad}^j(V)(W_i)) \oplus (\operatorname{ad}^{j+1}(V)(W)), (\partial_K)_0 \right)$$

in which each summand $((\oplus_i \operatorname{ad}^j(V)(W_i)) \oplus (\operatorname{ad}^{j+1}(V)(W)), (\partial_K)_0)$ is an acyclic DG vector space. It is now evident that $H(Q(K), (\partial_K)_0) \cong W$ and that the linearization of p' , that is, $(p')_0: (Q(K), (\partial_K)_0) \rightarrow (W, \partial_0 = 0)$, is a quasi-isomorphism of DG vector spaces. By [3, Proposition 22.12], p' is a quasi-isomorphism of DG Lie algebras.

In (1), left and right vertical arrows are now quasi-isomorphisms. Therefore, by passing to homology and applying the five-lemma, we obtain that p is a quasi-isomorphism. \square

Since it is the main case we require here, we write out explicitly what this gives for the model of $S^n \times X$, with a slight easing of notation.

Corollary 2.2. *Let X be a simply connected CW complex of finite type with Quillen minimal model $\mathbb{L}(W; d_X)$. Let $\mathbb{L}(v)$ with $|v| = n - 1$ and zero differential be the Quillen model of S^n , and set $W' = s^n(W)$. Let $\lambda: \mathbb{L}(W) \rightarrow \mathbb{L}(W, v, W')$ be the inclusion, and $S: \mathbb{L}(W) \rightarrow \mathbb{L}(W, v, W')$ be the λ -derivation that extends the linear map $S(w) = w'$. Define a differential ∂ on $\mathbb{L}(W, v, W')$ by $\partial(w) = d_X(w)$, $\partial(v) = 0$, and for each $w \in W$*

$$\partial(w') = (-1)^{n-1} [v, w] + (-1)^n S(d_X(w)).$$

Then $\mathbb{L}(W, v, W'; \partial)$ is the Quillen minimal model of $S^n \times X$. \square

3. Lie derivations and homotopy groups of function spaces

Say two maps of vector spaces $f: U \rightarrow V$ and $g: U' \rightarrow V'$ are *equivalent* if there exist isomorphisms α and β for which $\beta \circ f = g \circ \alpha$. We extend this notion of equivalence in the obvious way to exact sequences of vector spaces and any other diagram of vector space maps. Given any map $f: X \rightarrow Y$, we have the homomorphism

$$j_{\#} \otimes 1: \pi_n(\operatorname{map}_*(X, Y; f)) \otimes \mathbb{Q} \rightarrow \pi_n(\operatorname{map}(X, Y; f)) \otimes \mathbb{Q}$$

induced on rational homotopy groups by the fibre inclusion of the general evaluation fibration $\operatorname{map}_*(X, Y; f) \xrightarrow{j} \operatorname{map}(X, Y; f) \xrightarrow{\omega} Y$ hand, we have the homomorphism

$$H(J): H_n(\operatorname{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)) \rightarrow H_n(\operatorname{Rel}(\operatorname{ad}_{\mathcal{L}_f}))$$

that forms part of the long exact derivation homology sequence of the Quillen minimal model of f . In Theorem 3.1, we establish that these two homomorphisms are equivalent. This result and one immediate consequence will occupy the remainder of this section.

The main step is to establish vector space isomorphisms

$$\Phi: \pi_n(\operatorname{map}_*(X, Y; f)) \otimes \mathbb{Q} \rightarrow H_n(\operatorname{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f))$$

$$\Psi: \pi_n(\operatorname{map}(X, Y; f)) \otimes \mathbb{Q} \rightarrow H_n(\operatorname{Rel}(\operatorname{ad}_{\mathcal{L}_f})),$$

that give the equivalence. To this end, we define group homomorphisms

$$\Phi': \pi_n(\operatorname{map}_*(X, Y; f)) \rightarrow H_n(\operatorname{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f))$$

$$\Psi': \pi_n(\operatorname{map}(X, Y; f)) \rightarrow H_n(\operatorname{Rel}(\operatorname{ad}_{\mathcal{L}_f}))$$

for $n \geq 2$. Then the isomorphisms Φ and Ψ are obtained as the rationalizations of these homomorphisms.

In the following, we assume a fixed choice of Quillen minimal model $\mathcal{L}_f: \mathcal{L}_X \rightarrow \mathcal{L}_Y$. Write $\mathcal{L}_X = \mathbb{L}(W; d_X)$ and $\mathcal{L}_Y = \mathbb{L}(V; d_Y)$. Now define Φ' as follows. Let $\alpha \in \pi_n(\operatorname{map}_*(X, Y; f))$ be represented by a map $a: S^n \rightarrow \operatorname{map}_*(X, Y; f)$. Then the adjoint $A: S^n \times X \rightarrow Y$ of a has Quillen minimal model $\mathcal{L}_A: \mathcal{L}_{S^n \times X} \rightarrow \mathcal{L}_Y$. From Corollary 2.2 we have $\mathcal{L}_{S^n \times X} = \mathbb{L}(W, v, W'; \partial)$. Since a is a (based) map into the function space of *based* maps, A is a map *under* $(* \mid f): S^n \vee X \rightarrow Y$. (See Appendix A for this and other terminology we use in the course of this proof.) From Proposition A.3, we may take the Quillen minimal model of A to be a DG Lie algebra map

$\mathcal{L}_A: \mathbb{L}(W, v, W'; \partial) \rightarrow \mathcal{L}_Y$ under $\mathcal{L}_{(*|f)}: \mathbb{L}(W, v) \rightarrow \mathcal{L}_Y$, that is, such that $\mathcal{L}_A(v) = 0$ and $\mathcal{L}_A(w) = \mathcal{L}_f(w)$ for each $w \in W$. Now define a linear map $\theta_A: \mathbb{L}(W) \rightarrow \mathcal{L}_Y$ that increases degree by n as the composition

$$\mathbb{L}(W) \xrightarrow{S} \mathbb{L}(W, v, W') \xrightarrow{\mathcal{L}_A} \mathcal{L}_Y,$$

where $S: \mathbb{L}(W) \rightarrow \mathbb{L}(W, v, W')$ is the derivation from Corollary 2.2. A straightforward check shows that θ_A is an \mathcal{L}_f -derivation and a cycle in $\text{Der}_n(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$. We set $\Phi'(\alpha) = \langle \theta_A \rangle$.

We define Ψ' , and thus its rationalization Ψ , in a similar manner. Here, the adjoint $A: S^n \times X \rightarrow Y$ of $\alpha: S^n \rightarrow \text{map}(X, Y; f)$ still satisfies $A \circ i_2 = f: X \rightarrow Y$, but the composition $A \circ i_1: S^n \rightarrow Y$ may give a non-trivial element of $\pi_n(Y)$. By Proposition A.3, the Quillen minimal model of A satisfies $\mathcal{L}_A(w) = \mathcal{L}_f(w)$ for each $w \in W$, but $\mathcal{L}_A(v) \in \mathcal{L}_Y$ is now some non-trivial d_Y -cycle. As before, setting $\theta_A = \mathcal{L}_A \circ S: \mathbb{L}(W) \rightarrow \mathcal{L}_Y$ defines an \mathcal{L}_f -derivation in $\text{Der}_n(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$. It is easy to check that $((-1)^n \mathcal{L}_A(v), \theta_A) \in \text{Rel}_n(\text{ad}_{\mathcal{L}_f})$ is a cycle of the mapping cone. We set $\Psi'(\alpha) = \langle (-1)^n \mathcal{L}_A(v), \theta_A \rangle$.

Theorem 3.1. *Let $f: X \rightarrow Y$ be a map between simply connected CW complexes with X finite. Then we have:*

- (A) Φ' and Ψ' are well-defined homomorphisms;
- (B) Their rationalizations Φ and Ψ are isomorphisms;
- (C) The following square is commutative ($n \geq 2$):

$$\begin{array}{ccc} \pi_n(\text{map}_*(X, Y; f)) \otimes \mathbb{Q} & \xrightarrow[\cong]{\Phi} & H_n(\text{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)) \\ j_{\#} \otimes 1 \downarrow & & \downarrow H(J) \\ \pi_n(\text{map}(X, Y; f)) \otimes \mathbb{Q} & \xrightarrow[\Psi]{\cong} & H_n(\text{Rel}(\text{ad}_{\mathcal{L}_f})) \end{array}$$

Proof. Throughout the proof we will give full details for arguments concerning Φ ; the arguments for Ψ are similar and we will only indicate them briefly. Appendix A contains a careful justification of some technical details about DG Lie algebra homotopy theory used in the following proof. We will make free use of the material from Appendix A.

(A) Φ' is well-defined: Suppose that $a, b: S^n \rightarrow \text{map}_*(X, Y; f)$ are homotopic representatives of the homotopy class α . The adjoint of the homotopy from a to b gives a homotopy of their adjoints $A, B: S^n \times X \rightarrow Y$. Because the homotopy from a to b is a *based* homotopy, A and B are *homotopic under* $(* | f): S^n \vee X \rightarrow Y$. By Proposition A.3, the Quillen minimal models $\mathcal{L}_A, \mathcal{L}_B: \mathcal{L}_{S^n \times X} \rightarrow \mathcal{L}_Y$ are DG Lie *homotopic under* $\mathcal{L}_{(*|f)}: \mathcal{L}_{S^n \vee X} \rightarrow \mathcal{L}_Y$. This means that, with respect to the relative cylinder $c: \mathbb{L}(W, v, W'_1, W'_2; \partial) \rightarrow \mathbb{L}(W, v, W', sW', \widehat{W}'; \partial_I)$ for the inclusion $\mathbb{L}(W, v) \rightarrow \mathbb{L}(W, v, W'; \partial)$ described in Example A.1, we have a DG Lie algebra homotopy $\mathcal{H}: \mathbb{L}(W, v, W', sW', \widehat{W}'; D) \rightarrow \mathcal{L}_Y$ that satisfies $\mathcal{H} \circ c = (\mathcal{L}_A | \mathcal{L}_B)$. Let $j: \mathbb{L}(W; d_X) \rightarrow \mathbb{L}(W, v, W'_1, W'_2; \partial)$ denote the inclusion and let $S_1: \mathbb{L}(W) \rightarrow \mathbb{L}(W, v, W'_1, W'_2)$ denote the j -derivation defined on generators by $S_1(w) = w'_1$. Now define a linear map $\Theta: \mathbb{L}(W) \rightarrow \mathcal{L}_Y$ of degree $n+1$ as the composition $\Theta = \mathcal{H} \circ \sigma \circ c \circ S_1$. Here, σ is the derivation of degree $+1$ of $\mathbb{L}(W, v, W', sW', \widehat{W}')$ defined in Example A.1. Because $\mathcal{H} \circ c = \mathcal{L}_f$ on $\mathbb{L}(W)$, Θ is an \mathcal{L}_f -derivation. In Lemma A.2 we show that $c(w'_2) = w' + \widehat{w}' + \sigma \circ \partial_I(w')$. With this identity, we may compute as follows:

$$\begin{aligned} \theta_B(w) &= \mathcal{L}_B(w') = (\mathcal{L}_A | \mathcal{L}_B)(w'_2) = \mathcal{H} \circ c(w'_2) = \mathcal{H}(w' + \widehat{w}' + \sigma \circ \partial_I(w')) \\ &= \mathcal{H}(w') + \mathcal{H}(\partial_I(\sigma(w'))) + \mathcal{H} \circ \sigma((-1)^{n-1}[v, w] + (-1)^n S_I \partial_I(w)). \end{aligned}$$

Now $w' = c \circ S_1(w)$ and $S_I \partial_I(w) = c \circ S_1(d_X(w))$. Since σ is zero on all generators other than W' , and \mathcal{H} is a DG map, we may continue:

$$\begin{aligned} \theta_B(w) &= \mathcal{H} \circ c \circ S_1(w) + d_Y(\mathcal{H} \circ \sigma \circ c \circ S_1(w)) + (-1)^n \mathcal{H} \circ \sigma \circ c \circ S_1(d_X(w)) \\ &= \theta_A(w) + d_Y(\Theta(w)) + (-1)^n \Theta(d_X(w)). \end{aligned}$$

That is, we have $D\Theta = \theta_B - \theta_A \in \text{Der}_n(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$ and Φ' is well-defined.

Φ' is a homomorphism: Let $v: S^n \rightarrow S^n \vee S^n$ denote the usual pinching comultiplication. For $\alpha, \beta \in \pi_n(\text{map}_*(X, Y; f))$, we have $\alpha + \beta = (\alpha | \beta) \circ v$. Suppose α, β have adjoints $A, B: S^n \times X \rightarrow Y$, respectively.

Let $i_1, i_2: S^n \rightarrow S^n \vee S^n$ denote the inclusions, and let $(A \mid B)_f: (S^n \vee S^n) \times X \rightarrow Y$ be the map defined by $(A \mid B)_f \circ (i_1 \times 1) = A$ and $(A \mid B)_f \circ (i_2 \times 1) = B$. Then the adjoint of $\alpha + \beta$ is $C := (A \mid B)_f \circ (v \times 1): S^n \times X \rightarrow Y$. We focus on identifying the Quillen minimal model of $(A \mid B)_f$, and it will follow that Φ'_f is a homomorphism.

Since $(A \mid B)_f$ is a map under $(\ast \mid \ast \mid f): S^n \vee S^n \vee X \rightarrow Y$, we may combine Theorem 2.1 and Proposition A.3 and write the Quillen minimal model of $(A \mid B)_f$ in the form $\Gamma: \mathbb{L}(W, v_1, v_2, W_1, W_2; \partial) \rightarrow \mathcal{L}_Y$ with $\Gamma(v_1) = 0 = \Gamma(v_2)$ and such that Γ restricts to \mathcal{L}_f on $\mathbb{L}(W)$. Now the restriction of Γ to $\mathbb{L}(W, v_1, W_1; \partial)$ is a Quillen minimal model of A . Moreover, since the composition $(\alpha \mid \beta) \circ i_1$ is determined up to based homotopy as α , the composition $(A \mid B)_f \circ (i_1 \times 1): S^n \times X \rightarrow Y$ is determined up to a homotopy under $(\ast \mid f)$ as A . Therefore, the restriction of Γ to $\mathbb{L}(W, v_1, W_1; \partial)$ is actually DG Lie homotopic as a map under $\mathcal{L}_{(\ast \mid f)}$ to \mathcal{L}_A , by Proposition A.3. Following the argument given above to show that Φ' is well-defined, we see that $\langle \Gamma \circ S_1 \rangle = \Phi'(\alpha)$. Here, $S_1: \mathbb{L}(W) \rightarrow \mathbb{L}(W, v_1, v_2, W_1, W_2)$ denotes the derivation, relative to the inclusion, defined by $S_1(w) = w_1$ for each $w \in W$ as in Section 2. A similar argument results in an identification $\langle \Gamma \circ S_2 \rangle = \Phi'(\beta)$. Finally, since $v: S^n \rightarrow S^n \vee S^n$ has Quillen model $\mathcal{L}_v: \mathbb{L}(v) \rightarrow \mathbb{L}(v_1, v_2)$ given by $\mathcal{L}_v(v) = v_1 + v_2$, it follows that $v \times 1: S^n \times X \rightarrow (S^n \vee S^n) \times X$ has Quillen model $\mathcal{L}_{v \times 1}: \mathbb{L}(W, v, W'; \partial) \rightarrow \mathbb{L}(W, v_1, v_2, W_1, W_2; \partial)$ that satisfies $\mathcal{L}_{v \times 1}(w') = w_1 + w_2$. Thus we have $\langle \mathcal{L}_C(w') \rangle = \langle \Gamma \circ \mathcal{L}_{v \times 1}(w') \rangle = \langle \Gamma(w_1 + w_2) \rangle = \langle \Gamma(S_1(w) + S_2(w)) \rangle = \langle \theta_A(w) \rangle + \langle \theta_B(w) \rangle$. It follows that $\Phi'(\alpha + \beta) = \Phi'(\alpha) + \Phi'(\beta)$, that is, Φ' is a homomorphism.

Ψ' is a well-defined homomorphism: This is established by making small adjustments to the preceding arguments for Φ' . In this case, the homotopy of the adjoints is stationary at f on X , but is not necessarily stationary on S^n . Therefore, we adjust the relative cylinder to that for the Quillen minimal model of the inclusion $i_2: X \rightarrow S^n \times X$, namely

$$c: \mathbb{L}(W, v_1, v_2, W'_1, W'_2; \partial) \rightarrow \mathbb{L}(W, v, sv, \widehat{v}, W', sW', \widehat{W}'; \partial_I).$$

Since $\partial_I(v) = 0$, we have $c(v_2) = v + \widehat{v}$ and hence $\mathcal{L}_B(v) = (\mathcal{L}_A \mid \mathcal{L}_B)(v_2) = \mathcal{H} \circ c(v_2) = \mathcal{H}(v) + \mathcal{H}(\partial_I(sv)) = \mathcal{L}_A(v) + d_Y \mathcal{H}(sv)$. A careful check – use (7) and the remark that follows it – reveals that the formula of Lemma A.2 remains valid for this cylinder. We define an \mathcal{L}_f -derivation $\Theta = \mathcal{H} \circ \sigma \circ c \circ S_1$ and proceed as before.

(B) *Φ is a surjection:* Suppose given $\theta \in \text{Der}_n(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$, a cycle derivation of degree n . Define a Lie algebra map $\mathcal{L}_A: \mathbb{L}(W, v, W'; \partial) \rightarrow \mathcal{L}_Y$ by setting $\mathcal{L}_A(w) = \mathcal{L}_f(w)$, $\mathcal{L}_A(v) = 0$ and $\mathcal{L}_A(w') = \theta(w)$ for $w \in W$. Just as in the definition of Φ' , $\mathcal{L}_A \circ S$ is an \mathcal{L}_f -derivation and by construction we have $\mathcal{L}_A \circ S = \theta$. It is straightforward to check that \mathcal{L}_A commutes with differentials. Let $A: S_{\mathbb{Q}}^n \times X_{\mathbb{Q}} \rightarrow Y_{\mathbb{Q}}$ be the geometric realization of \mathcal{L}_A , from the correspondence between (homotopy classes of) maps between rational spaces and DG Lie algebra maps between Quillen models. Let $i_1: S_{\mathbb{Q}}^n \rightarrow S_{\mathbb{Q}}^n \times X_{\mathbb{Q}}$ and $i_2: X_{\mathbb{Q}} \rightarrow S_{\mathbb{Q}}^n \times X_{\mathbb{Q}}$ denote the inclusions. Since $\mathcal{L}_A \circ \mathcal{L}_{i_1} = 0$ and $\mathcal{L}_A \circ \mathcal{L}_{i_2} = \mathcal{L}_f$, we have $A \circ i_1 \sim \ast$ and $A \circ i_2 \sim f_{\mathbb{Q}}$. Altering the geometric realization A up to homotopy, if necessary, we may assume $A \circ i_1 = \ast$ and $A \circ i_2 = f_{\mathbb{Q}}$. Thus, the adjoint $a: S_{\mathbb{Q}}^n \rightarrow \text{map}_*(X_{\mathbb{Q}}, Y_{\mathbb{Q}}; f_{\mathbb{Q}})$ of A represents an element $\alpha \in \pi_n(\text{map}_*(X_{\mathbb{Q}}, Y_{\mathbb{Q}}; f_{\mathbb{Q}}))$. Since X is finite, [6, Th. II.3.11] and [11, Th. 2.3], imply $\pi_n(\text{map}(X, Y; f)) \otimes \mathbb{Q} \cong \pi_n(\text{map}(X_{\mathbb{Q}}, Y_{\mathbb{Q}}; f_{\mathbb{Q}}))$. By definition, $\Phi(\alpha) = \langle \theta \rangle$, and so Φ is surjective.

Φ is an injection: Since Φ is a homomorphism, it is sufficient to check that $\Phi(\alpha) = 0$ implies $\alpha = 0 \in \pi_n(\text{map}_*(X, Y_{\mathbb{Q}}; f_{\mathbb{Q}}))$. As before, write $\Phi(\alpha) = \langle \theta_A \rangle$ and suppose $\theta_A \in \text{Der}_n(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$ is a boundary so that $\theta_A = D(\Theta)$ for $\Theta \in \text{Der}_{n+1}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$. We refer to the relative cylinder described in Example A.1. Define a relative homotopy $\mathcal{G}: \mathbb{L}(W, v, W', sW', \widehat{W}'; \partial_I) \rightarrow \mathcal{L}_Y$ by setting $\mathcal{G} = \mathcal{L}_A$ on $\mathbb{L}(W, v, W')$ (so that \mathcal{G} starts at \mathcal{L}_A). Then set $\mathcal{G}(sw') = -\Theta(w)$ and $\mathcal{G}(\widehat{w}') = -d_Y \Theta(w)$ for $w \in W$ and extend \mathcal{G} as a Lie algebra map. It is automatic from this definition that \mathcal{G} is a DG map. Since $\sigma = 0$ on all generators other than those of W' , it follows that $\mathcal{G} \circ \sigma \circ S_I$ acts as the \mathcal{L}_f -derivation $-\Theta$ in $\text{Der}_{n+1}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$. With the formula from Lemma A.2, we compute as follows:

$$\begin{aligned} \mathcal{G} \circ c(w'_2) &= \mathcal{G}(w' + \widehat{w}' + \sigma(\partial_I(w'))) \\ &= \mathcal{L}_A(w') - d_Y \Theta(w) + \mathcal{G}(\sigma((-1)^{n-1}[v, w] + (-1)^n S_I d_X(w))) \\ &= \mathcal{L}_A(w') - d_Y \Theta(w) + (-1)^n \mathcal{G}(\sigma(S_I d_X(w))) \\ &= \mathcal{L}_A(w') - d_Y \Theta(w) - (-1)^n \Theta(d_X(w)) = \mathcal{L}_A(w') - D(\Theta)(w) = 0. \end{aligned}$$

Hence we have $\mathcal{G} \circ c = (\mathcal{L}_A \mid \mathcal{L}_{f \circ p_2})$, where $p_2: S^n \times X \rightarrow X$ denotes projection onto the second factor and $\mathcal{L}_{f \circ p_2}$ denotes the Quillen minimal model of the composition $f \circ p_2: S^n \times X \rightarrow Y$. It follows that $A \sim f \circ p_2: S^n \times X \rightarrow Y_{\mathbb{Q}}$. Taking adjoints, we obtain that $a \sim \ast: S^n \rightarrow \text{map}_*(X, Y; f)$. Actually, we only obtain this last homotopy as a free homotopy by taking adjoints, since the homotopy between A and $f \circ p_2$ is based, but not necessarily under

$(* \mid f): S^n \vee X \rightarrow Y$. However, a based map from a sphere is based-homotopic to the constant map if it is freely homotopic to the constant map [12, p. 27]. Thus Φ is injective.

Ψ is an isomorphism: Once again, we need only make slight adjustments to the preceding arguments for Φ . Suppose given (y, θ) a δ -cycle in $\text{Rel}_n(\text{ad}_{\mathcal{L}_f})$. Then $d_Y(y) = 0$ and $D(\theta) + \text{ad}_{\mathcal{L}_f}(y) = 0$, or $d_Y\theta(\chi) - (-1)^n\theta d_X(\chi) + [y, \mathcal{L}_f(\chi)] = 0$ for $\chi \in \mathcal{L}_X$. Now define $\mathcal{L}_A: \mathbb{L}(W, v, W'; \partial) \rightarrow \mathcal{L}_Y$ by $\mathcal{L}_A(w) = \mathcal{L}_f(w)$, $\mathcal{L}_A(v) = (-1)^n y$, and $\mathcal{L}_A(w') = \theta(w)$. Check that \mathcal{L}_A is a DG map and then argue that Ψ is surjective following the same steps as were taken for Φ .

To show injectivity of Ψ , suppose that $((-1)^n \mathcal{L}_A(v), \theta_A) \in \text{Rel}_n(\text{ad}_{\mathcal{L}_f})$ is a boundary in the mapping cone, so that $((-1)^n \mathcal{L}_A(v), \theta_A) = \delta(y, \theta)$. That is, $\theta_A = \text{ad}_{\mathcal{L}_f}(y) + d_Y \theta - (-1)^{n+1} \theta d_X$ and $d_Y(y) = (-1)^{n-1} \mathcal{L}_A(v)$. Here, we use the same relative mapping cylinder as was used to show Ψ' a homomorphism above. Define a DG homotopy by setting $\mathcal{G} = \mathcal{L}_A$ on $\mathbb{L}(W, v, W')$, $\mathcal{G}(sv) = (-1)^n y$, $\mathcal{G}(\widehat{v}) = -\mathcal{L}_A(v)$, $\mathcal{G}(sw') = -\theta(w)$, and $\mathcal{G}(\widehat{w}') = -d_Y \theta(w)$. Argue as for Φ to show that Ψ is injective.

(C) The commutativity of the diagram follows directly from the definitions. Notice that $j: \text{map}_*(X, Y; f) \rightarrow \text{map}(X, Y; f)$ is the fibre inclusion of the evaluation fibration. Therefore, if a is a representative of $\alpha \in \pi_n(\text{map}_*(X, Y; f))$, we may also take a to be a representative of $j_\#(\alpha) \in \pi_n(\text{map}(X, Y; f))$. Consequently, the Quillen minimal model of the adjoint of a representative of $j_\#(\alpha)$ may be taken as \mathcal{L}_A with $\mathcal{L}_A(v) = 0$. Hence, before rationalizing, we have $\Psi'(j_\#(\alpha)) = \langle 0, \theta_A \rangle = \langle J(\theta_A) \rangle = H(J)(\langle \theta_A \rangle) = H(J) \circ \Phi'(\alpha)$. \square

We deduce some immediate consequences of this result. To begin, we obtain a description of the long exact rational homotopy sequence of the evaluation fibration $\text{map}_*(X, Y; f) \xrightarrow{j} \text{map}(X, Y; f) \xrightarrow{\omega} Y$ for a map $f: X \rightarrow Y$. Stepping back in the Barratt–Puppe sequence, we obtain a fibration

$$\Omega Y \xrightarrow{\partial} \text{map}_*(X, Y; f) \xrightarrow{j} \text{map}(X, Y; f)$$

and thus a long exact sequence in homotopy

$$\cdots \xrightarrow{\partial_\#} \pi_n(\text{map}_*(X, Y; f)) \xrightarrow{j_\#} \pi_n(\text{map}(X, Y; f)) \xrightarrow{\omega_\#} \pi_{n-1}(\Omega Y) \xrightarrow{\partial_\#} \cdots \quad (2)$$

Here, we are identifying $\pi_{n-1}(\Omega Y)$ with $\pi_n(Y)$ in the usual way.

Theorem 3.2. *Let $f: X \rightarrow Y$ be a map between simply connected CW complexes of finite type with X finite. Then the rationalization of the long exact homotopy sequence (2), as far as the term $\pi_1(\Omega Y) \otimes \mathbb{Q}$, is equivalent to the long exact derivation homology sequence of the Quillen minimal model $\mathcal{L}_f: \mathcal{L}_X \rightarrow \mathcal{L}_Y$ of f , that is,*

$$\cdots \xrightarrow{H(\text{ad}_{\mathcal{L}_f})} H_n(\text{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)) \xrightarrow{H(J)} H_n(\text{Rel}(\text{ad}_{\mathcal{L}_f})) \xrightarrow{H(P)} H_{n-1}(\mathcal{L}_Y) \xrightarrow{H(\text{ad}_{\mathcal{L}_f})} \cdots$$

as far as the term $H_1(\mathcal{L}_Y)$.

Proof. The result follows from Theorem 3.1 and the uniqueness of the “third rung” in a commutative ladder (see [8, Lem. 3.1]) for $n \geq 2$. Checking the result at the last square is straightforward using the ideas of the proof of Theorem 3.1. \square

Theorem 3.3. *Let $f: X \rightarrow Y$ be a map between simply connected CW complexes of finite type with X finite. Then, for $n \geq 2$, we have*

$$G_n(Y, X; f) \otimes \mathbb{Q} \cong \ker\{H(\text{ad}_{\mathcal{L}_f}): H_{n-1}(\mathcal{L}_Y) \rightarrow H_{n-1}(\text{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f))\}.$$

Proof. The group $G_n(Y, X; f) \otimes \mathbb{Q}$ corresponds to the kernel of $\partial: \pi_{n-1}(\Omega Y) \otimes \mathbb{Q} \rightarrow \pi_{n-1}(\text{map}_*(X, Y; f)) \otimes \mathbb{Q}$ in the long exact rational homotopy sequence of the evaluation fibration. The result thus follows from Theorem 3.2. \square

Specializing to the identity map we recover a result of Tanré:

Corollary 3.4 ([13, Cor. VII.4(10)]). *Let X be a simply connected, finite complex. Then, for $n \geq 2$, we have*

$$G_n(X) \otimes \mathbb{Q} \cong \ker\{H(\text{ad}_{\mathcal{L}_X}): H_{n-1}(\mathcal{L}_X) \rightarrow H_{n-1}(\text{Der}(\mathcal{L}_X))\}. \quad \square$$

4. Evaluation subgroups and Gottlieb's question

Let $[\ , \]_w$ denote the Whitehead bracket in $\pi_*(X)$ and let $P_*(X)$ denote the subgroup of $\pi_*(X)$ consisting of homotopy elements with vanishing Whitehead product with all elements of $\pi_*(X)$ — the so-called *Whitehead center* of $\pi_*(X)$. It is well known that $G_*(X) \subseteq P_*(X)$; Gottlieb's question asks when is the inclusion strict? Ganea gave the first example of inequality in [4]. See [9] for a recent reference and some interesting examples of $G_1(X) \neq P_1(X)$ with X a finite complex.

Gottlieb's question admits the following generalization. Given a map $f: X \rightarrow Y$ set

$$P_n(Y, X; f) = \{\alpha \in \pi_n(Y) \mid [\alpha, f_*(\beta)]_w = 0 \text{ for all } \beta \in \pi_n(X)\}.$$

Then $G_n(Y, X; f)$ is a subgroup of $P_n(Y, X; f)$ and we may ask whether these groups can be different. We give a rather complete answer to this question for rational spaces. Notice that although the relative evaluation subgroup behaves well with respect to rationalization, in the sense that $G_*(Y, X; f) \otimes \mathbb{Q} = G_*(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}})$ (at least for X finite), the inclusion $P_*(X, Y; f) \otimes \mathbb{Q} \subseteq P_*(X_{\mathbb{Q}}, Y_{\mathbb{Q}}; f_{\mathbb{Q}})$ is usually strict.

We show that, rationally, the difference between the evaluation subgroup of a map and the generalized Whitehead center is governed by the “induced derivation” map

$$I: H_n(\text{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)) \rightarrow \text{Der}_n(H(\mathcal{L}_X), H(\mathcal{L}_Y); H(\mathcal{L}_f))$$

which we now introduce. A D -cycle $\theta \in \text{Der}_n(L, K; \psi)$ induces a map $H(\theta) \in \text{Der}_n(H(L), H(K); H(\psi))$ defined by $H(\theta)(\langle \xi \rangle) = \langle \theta(\xi) \rangle$ for ξ a cycle of L . If θ is a D -boundary then it carries cycles of L to boundaries of K . Thus we obtain a linear map $I: H_n(\text{Der}(L, K; \psi)) \rightarrow \text{Der}_n(H(L), H(K); H(\psi))$ given by $I(\langle \theta \rangle) = H(\theta)$ for θ a cycle of $\text{Der}_n(L, K; \psi)$.

Now consider the commutative diagram

$$\begin{array}{ccc} H_n(\mathcal{L}_Y) & \xrightarrow{H(\text{ad}_{\mathcal{L}_f})} & H_n(\text{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)) \\ & \searrow \text{ad}_{H(\mathcal{L}_f)} & \downarrow I \\ & & \text{Der}_n(H(\mathcal{L}_X), H(\mathcal{L}_Y); H(\mathcal{L}_f)). \end{array}$$

Theorem 4.1. *Let $f: X \rightarrow Y$ be a map between simply connected CW complexes of finite type with X a finite complex. For $n \geq 1$, we have*

$$\frac{P_{n+1}(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}})}{G_{n+1}(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}})} \cong \ker(I) \cap \text{im}(H(\text{ad}_{\mathcal{L}_f})).$$

Proof. The map $H(\text{ad}_{\mathcal{L}_f})$ induces a map

$$\frac{\ker(\text{ad}_{H(\mathcal{L}_f)})}{H(\text{ad}_{\mathcal{L}_f}): \ker(H(\text{ad}_{\mathcal{L}_f}))} \rightarrow \ker(I) \cap \text{im}(H(\text{ad}_{\mathcal{L}_f}))$$

that is easily checked to be an isomorphism. By Theorem 3.3, we may identify $\ker(H(\text{ad}_{\mathcal{L}_f}))$ with $G_{n+1}(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}})$. We may also identify $\ker(\text{ad}_{H(\mathcal{L}_f)})$ with $P_{n+1}(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}})$ by the correspondence between the Samelson product in $\pi_*(\Omega Y) \otimes \mathbb{Q}$ and the product on $H(\mathcal{L}_Y)$. It is easy to check that these identifications are compatible. \square

Example 4.2. We give an example with $G_*(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}}) \neq P_*(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}})$. Consider $f: \mathbb{C}P^2 \rightarrow S^4$ obtained by pinching out the 2-cell of $\mathbb{C}P^2$. This map has Quillen minimal model

$$\mathcal{L}_f: \mathbb{L}(x_1, x_3; d_X) \rightarrow \mathbb{L}(u_3; d_Y = 0)$$

with $\mathcal{L}_f(x_1) = 0$ and $\mathcal{L}_f(x_3) = u_3$. Here, the subscript of a generator denotes its degree and $d_X(x_1) = 0$, $d_X(x_3) = [x_1, x_1]$. Now $\text{ad}_{\mathcal{L}_f}(u_3) \in \text{Der}_3(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$ is defined by $\text{ad}_{\mathcal{L}_f}(u_3)(x_1) = 0$ and $\text{ad}_{\mathcal{L}_f}(u_3)(x_3) = [u_3, u_3]$. On the other hand, $D = 0$ in $\text{Der}_3(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$. Therefore, $H(\text{ad}_{\mathcal{L}_f})(\langle u_3 \rangle) \neq 0$ and $G_4(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}}) = 0$.

We next give a class of examples for which the equality $G_*(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}}) = P_*(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}})$ holds. For this, we review the notion of *coformality* and some terminology associated with this concept. Suppose that a minimal DG Lie algebra $\mathbb{L}(V; d)$ has a second (or “upper”) grading on the generating subspace $V = \bigoplus_{i \geq 0} V^i$. This extends to a second grading of $\mathbb{L}(V)$ in the obvious way, and we write $\mathbb{L}(V)^i$ for the sub-vector space of $\mathbb{L}(V)$ consisting of all elements of $\mathbb{L}(V)$ of second grading equal to i . We also write $V^{(i)}$ for the sub-vector space of V consisting of all elements of V of second grading less than or equal to i . Then we say that $\mathbb{L}(V; d)$ is a *bigraded* minimal DG Lie algebra if the differential decreases second degree homogeneously by one, that is, if $d(V^0) = 0$ and $d(V^i) \subseteq \mathbb{L}(V)^{i-1}$ for $i \geq 1$. If $\mathbb{L}(V; d)$ is a bigraded minimal DG Lie algebra, then the second grading passes to homology, making $H(\mathbb{L}(V; d))$ a bigraded Lie algebra. We write $H^i(\mathbb{L}(V; d))$ for the sub-vector space of $H(\mathbb{L}(V; d))$ consisting of homology classes represented by cycles of upper degree equal to i , and we have $H(\mathbb{L}(V; d)) = \bigoplus_{i \geq 0} H^i(\mathbb{L}(V; d))$.

Definition 4.3. Let $\mathbb{L}(V; d)$ be a bigraded minimal DG Lie algebra in the above sense. We say $\mathbb{L}(V; d)$ is *coformal* if $H^i(\mathbb{L}(V; d)) = 0$ for $i > 0$, so that $H(\mathbb{L}(V; d)) = H^0(\mathbb{L}(V; d))$. We say that a space X is a *coformal space* if its Quillen minimal model is coformal.

There are many interesting examples of coformal spaces: Moore spaces and more generally rational co-H-spaces, including suspensions; some homogeneous spaces; products and wedges of coformal spaces. This notion of coformality extends to a map. Suppose that $\phi: \mathbb{L}(V; d) \rightarrow \mathbb{L}(W; d')$ is a map of bigraded minimal DG Lie algebras as defined above. If $\phi(V^i) \subseteq \mathbb{L}(W)^i$ for each $i \geq 0$, then we say that ϕ is a bigraded map.

Definition 4.4. A map $\phi: \mathbb{L}(V; d) \rightarrow \mathbb{L}(W; d')$ of bigraded minimal DG Lie algebras is a *coformal map* if both $\mathbb{L}(V; d)$ and $\mathbb{L}(W; d')$ are coformal, and ϕ is a bigraded map (with respect to the second gradings that display the coformality of $\mathbb{L}(V)$ and $\mathbb{L}(W)$). A map of coformal spaces $f: X \rightarrow Y$ is a *coformal map* if its Quillen minimal model $\mathcal{L}_f: \mathcal{L}_X \rightarrow \mathcal{L}_Y$ is a coformal map of bigraded minimal DG Lie algebras.

Theorem 4.5. Let $f: X \rightarrow Y$ be a coformal map between CW complexes of finite type with X finite. Then $P_*(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}}) = G_*(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}})$.

Proof. Suppose $\mathcal{L}_X = \mathbb{L}(W; d_X)$ and $\mathcal{L}_Y = \mathbb{L}(V; d_Y)$ are coformal, and that \mathcal{L}_f is bigraded. Take $\alpha \in H_n(\mathcal{L}_Y)$. With reference to Theorem 4.1, we show that if $I \circ H(\text{ad}_{\mathcal{L}_f})(\alpha) = 0$, then $H(\text{ad}_{\mathcal{L}_f})(\alpha) = 0$. Since Y is coformal, we may assume $\alpha = \langle \xi \rangle$ for a d_Y -cycle $\xi \in \mathbb{L}(V)^0$. Observe that $\text{ad}_{\mathcal{L}_f}(\xi) \in \text{Der}_n(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$ is then a D -cycle that preserves upper degree. If $I \circ H(\text{ad}_{\mathcal{L}_f})(\alpha) = 0$, then for each d_X -cycle $\chi \in \mathbb{L}(W)$, we have $\text{ad}_{\mathcal{L}_f}(\xi)(\chi) = d_Y(\eta)$ for some $\eta \in \mathbb{L}(V)$. We now use this to construct $\theta \in \text{Der}_{n+1}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$ such that $D(\theta) = \text{ad}_{\mathcal{L}_f}(\xi)$.

Since X is coformal, we have $W = \bigoplus_{i \geq 0} W^i$, and each $w \in W^0$ is a d_X -cycle. Therefore, we have $\text{ad}_{\mathcal{L}_f}(\xi)(w) = d_Y(\eta)$ for some $\eta \in \mathbb{L}(V)$. Furthermore, since \mathcal{L} is bigraded, we may choose $\eta \in \mathbb{L}(V)^1$. Use this to define a linear map $\theta_0: W^0 \rightarrow \mathbb{L}(V)^1$ and extend to an \mathcal{L}_f -derivation $\theta_0 \in \text{Der}_{n+1}(\mathbb{L}(W^0), \mathcal{L}_Y; \mathcal{L}_f)$. By construction, we have $D(\theta_0)(\chi) = d_Y(\theta_0(\chi)) = \text{ad}_{\mathcal{L}_f}(\xi)(\chi)$ for $\chi \in \mathbb{L}(W^0)$.

Assume inductively that $\theta_m \in \text{Der}_{n+1}(\mathbb{L}(W^{(m)}), \mathcal{L}_Y; \mathcal{L}_f)$ is defined, increasing upper degree homogeneously by 1, and satisfying $D(\theta_m) = \text{ad}_{\mathcal{L}_f}(\xi)$ on $\mathbb{L}(W^{(m)})$. For $w \in W^{m+1}$, consider $\text{ad}_{\mathcal{L}_f}(\xi)(w) + (-1)^{n+1}\theta_m(d_X w)$. Since $\text{ad}_{\mathcal{L}_f}(\xi)$ is a D -cycle, and $d_X w \in \mathbb{L}(W)^m \subseteq \mathbb{L}(W^{(m)})$, we see that this is a d_Y -cycle. Now use the coformality of \mathcal{L}_Y – specifically, that $H^+(\mathcal{L}_Y) = 0$ – to conclude that there exists some $\zeta \in \mathcal{L}_Y$ with $d_Y(\zeta) = \text{ad}_{\mathcal{L}_f}(\xi)(w) + (-1)^{n+1}\theta_m(d_X w)$. Furthermore, we may choose $\zeta \in \mathbb{L}(V)^{m+2}$. Clearly, ζ may be chosen so as to depend linearly on w . So use this to define a linear map $\theta_{m+1}: W^{m+1} \rightarrow \mathbb{L}(V)^{m+2}$ with $\theta_{m+1}(w) = \zeta$, and extend θ_m to an \mathcal{L}_f -derivation $\theta_{m+1} \in \text{Der}_{n+1}(\mathbb{L}(W^{(m+1)}), \mathcal{L}_Y; \mathcal{L}_f)$. By construction, we have

$$\begin{aligned} D\theta_{m+1}(w) &= d_Y\theta_{m+1}(w) - (-1)^{n+1}\theta_{m+1}(d_X w) \\ &= d_Y\zeta - (-1)^{n+1}\theta_m(d_X w) = \text{ad}_{\mathcal{L}_f}(\xi)(w) \end{aligned}$$

for $w \in W^{m+1}$; since $D(\theta_{m+1})$ is an \mathcal{L}_f -derivation, this gives $D(\theta_{m+1})(\chi) = \text{ad}_{\mathcal{L}_f}(\xi)(\chi)$ for $\chi \in \mathbb{L}(W^{(m+1)})$. This completes the induction and gives an \mathcal{L}_f -derivation $\theta \in \text{Der}_{n+1}(\mathbb{L}(W), \mathcal{L}_Y; \mathcal{L}_f)$ that satisfies $D(\theta) = \text{ad}_{\mathcal{L}_f}(\xi)$. The result follows. \square

The following special case is well known.

Corollary 4.6. *Let X be a simply connected, finite CW complex. If X is coformal, then $P_*(X_{\mathbb{Q}}) = G_*(X_{\mathbb{Q}})$. \square*

5. The rationalized G -sequence

The G -sequence of a map $f: X \rightarrow Y$ is a chain complex featuring the Gottlieb groups $G_n(X)$ and the evaluation subgroups $G_n(Y, X; f)$. The G -sequence may be constructed as follows. The map f induces a diagram

$$\begin{array}{ccc} \Omega X & \xrightarrow{\Omega f} & \Omega Y \\ \downarrow \partial & & \downarrow \partial \\ \text{map}_*(X, X; 1) & \xrightarrow{f_*} & \text{map}_*(X, Y; f), \end{array}$$

in which the vertical maps are the connecting maps arising from the evaluation fibrations $\omega: \text{map}(X, X; 1) \rightarrow X$ and $\omega: \text{map}(X, Y; f) \rightarrow Y$ as in Section 3. The maps Ωf and f_* lead to long exact homotopy sequences and the vertical maps give homomorphisms of corresponding terms, yielding a homotopy ladder in the usual way. Whenever we have such a ladder, with exact rows, there is an associated “kernel sequence”, that is, a sequence obtained by restricting the maps in the top row to the kernels of the vertical rungs. The G -sequence of the map f may be defined, with a shift in degree, as the kernel sequence of the above homotopy ladder. A portion of this construction is shown here:

$$\begin{array}{ccccccc} \cdots & \xrightarrow{p} & G_{n+1}(X) & \xrightarrow{(\Omega f)_\#} & G_{n+1}(Y, X; f) & \xrightarrow{j} & G_{n+1}^{\text{rel}}(Y, X; f) \xrightarrow{p} \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \xrightarrow{p} & \pi_n(\Omega X) & \xrightarrow{(\Omega f)_\#} & \pi_n(\Omega Y) & \xrightarrow{j} & \pi_n(\Omega f) \xrightarrow{p} \cdots \\ & & \downarrow \partial_\# & & \downarrow \partial_\# & & \downarrow \Delta \\ \cdots & \xrightarrow{\hat{p}} & \pi_n(\text{map}_*(X, X; 1)) & \xrightarrow{(f_*)_\#} & \pi_n(\text{map}_*(X, Y; f)) & \xrightarrow{\hat{j}} & \pi_n(f_*) \xrightarrow{\hat{p}} \cdots \end{array}$$

Note that the maps in the G -sequence are just the restrictions of the maps in the long exact homotopy sequence of the map $\Omega f: \Omega X \rightarrow \Omega Y$. Thus compositions of consecutive maps in the G -sequence are trivial. However, the kernel sequence of a commutative ladder of exact sequences need not be exact, and so the G -sequence is a chain complex (of \mathbb{Z} -modules). The original description given in [14,7] (see also [8, Section 1]) represents the G -sequence as an image sequence, in a way obviously equivalent to the above.

We imitate the above construction in the framework of Lie derivations spaces and adjoints introduced in Section 1. Begin with the commutative square

$$\begin{array}{ccc} L & \xrightarrow{\quad} & K \\ \downarrow \text{ad} & & \downarrow \text{ad} \\ \text{Der}(L, L; 1) & \xrightarrow{\psi_*} & \text{Der}(L, K; \psi) \end{array}$$

of DG vector spaces. We then obtain the following commutative ladder of long exact homology sequences:

$$\begin{array}{ccccccc} \cdots & \xrightarrow{H(P)} & H_n(L) & \xrightarrow{H(\psi)} & H_n(K) & \xrightarrow{H(J)} & H_n(\text{Rel}(\psi)) \xrightarrow{H(P)} \cdots \\ & & \downarrow H(\text{ad}) & & \downarrow H(\text{ad}_\psi) & & \downarrow H(\text{ad}, \text{ad}_\psi) \\ \cdots & \xrightarrow{H(\hat{P})} & H_n(\text{Der}(L, L; 1)) & \xrightarrow{H(\psi_*)} & H_n(\text{Der}(L, K; \psi)) & \xrightarrow{H(\hat{J})} & H_n(\text{Rel}(\psi_*)) \xrightarrow{H(\hat{P})} \cdots \end{array} \quad (3)$$

To obtain unambiguous notation, we have written $\hat{J}: \text{Der}_n(L, K; \psi) \rightarrow \text{Rel}_n(\psi_*)$ and $\hat{P}: \text{Rel}_n(\psi_*) \rightarrow \text{Der}_{n-1}(L)$ for the usual inclusion and projection maps in the lower sequence.

Definition 5.1. Let $\psi: L \rightarrow K$ be a DG Lie algebra map. The n th evaluation subgroup of ψ is the subgroup

$$G_n(K, L; \psi) = \ker\{H(\text{ad}_\psi): H_{n-1}(K) \rightarrow H_{n-1}(\text{Der}(L, K; \psi))\}$$

of $H_{n-1}(K)$. The n th Gottlieb group (L, d) is the subgroup

$$G_n(L) = \ker\{H(\text{ad}): H_{n-1}(L) \rightarrow H_{n-1}(\text{Der}(L))\}$$

of $H_{n-1}(L)$. The n th relative evaluation subgroup of ψ $G_n^{\text{rel}}(K, L; \psi)$ is the subgroup

$$G_n^{\text{rel}}(K, L; \psi) = \ker\{H(\text{ad}, \text{ad}_\psi): H_{n-1}(\text{Rel}(\psi)) \rightarrow H_{n-1}(\text{Rel}(\psi_*))\}$$

of $H_{n-1}(\text{Rel}(\psi))$. The G -sequence of ψ is the sequence of kernels from the commutative ladder (3). That is, the sequence

$$\cdots \xrightarrow{H(P)} G_n(L) \xrightarrow{H(\psi)} G_n(K, L; \psi) \xrightarrow{H(J)} G_n^{\text{rel}}(K, L; \psi) \xrightarrow{H(P)} \cdots.$$

Theorem 5.2. *Let $f: X \rightarrow Y$ be a map between simply connected CW complexes of finite type, with X finite. Then the rationalization of the G -sequence of f*

$$\cdots \longrightarrow G_n(X) \xrightarrow{(\Omega f)_\# \otimes 1} G_n(Y, X; f) \otimes \mathbb{Q} \xrightarrow{j} G_n^{\text{rel}}(Y, X; f) \otimes \mathbb{Q} \longrightarrow \cdots$$

down to the term $G_3^{\text{rel}}(Y, X; f) \otimes \mathbb{Q}$ is equivalent to the G -sequence of the Quillen model $\mathcal{L}_f: \mathcal{L}_X \rightarrow \mathcal{L}_Y$ of f ,

$$\cdots \longrightarrow G_n(\mathcal{L}_X) \xrightarrow{H(\mathcal{L}_f)} G_n(\mathcal{L}_Y, \mathcal{L}_X; \mathcal{L}_f) \xrightarrow{H(J)} G_n^{\text{rel}}(\mathcal{L}_Y, \mathcal{L}_X; \mathcal{L}_f) \longrightarrow \cdots$$

down to the term $G_3^{\text{rel}}(\mathcal{L}_Y, \mathcal{L}_X; \mathcal{L}_f)$.

Proof. The result is a consequence of Theorem 3.2 but some care must be taken due to the non-natural choices involved in identifying the long exact homotopy sequence of the evaluation fibration. The argument needed to overcome this difficulty is the same as that used to deduce the corresponding result, Theorem 3.5, of [8]. \square

Remark 5.3. Under our hypotheses, $G_2(X) \otimes \mathbb{Q} = 0$, and there seems little to be gained by trying to extend Theorem 5.2 beyond this point.

As an application of the above, we consider the question of exactness of the G -sequence for cellular extensions. We focus on the rationalized G -sequence at the $G_n(X)$ term. Following Lee and Woo [7], define the ω -homology of a map $f: X \rightarrow Y$ at this term by setting

$$H_n^{a\omega}(Y, X; f) = \frac{\ker\{f_\#: G_n(X) \rightarrow G_n(Y, X; f)\}}{\text{im}\{p: G_{n+1}^{\text{rel}}(Y, X; f) \rightarrow G_n(X)\}}.$$

For a single cell-attachment, the following is a complete result for the lowest degree in which the (rationalized) G -sequence can be non-exact at the $G_n(X)$ -term.

Theorem 5.4. *Let X be a simply connected finite complex and $Y = X \cup_\alpha e^{n+1}$ for some $\alpha \in \pi_n(X)$. Suppose the following three conditions hold:*

- (1) $\alpha_{\mathbb{Q}} \neq 0$;
- (2) $\alpha_{\mathbb{Q}} \in G_n(X_{\mathbb{Q}})$;
- (3) Y is not rationally equivalent to a point.

Then the G -sequence of the inclusion $i: X \rightarrow Y$ is non-exact at the $G_n(X)$ term; indeed, we have $H_n^{a\omega}(Y, X; i) \otimes \mathbb{Q} = \mathbb{Q}$. Conversely, if any of (1)–(3) do not hold, then $H_n^{a\omega}(Y, X; i) \otimes \mathbb{Q} = 0$, that is,

$$G_{n+1}^{\text{rel}}(Y, X; i) \otimes \mathbb{Q} \xrightarrow{p \otimes 1} G_n(X) \otimes \mathbb{Q} \xrightarrow{i_\# \otimes 1} G_n(Y, X; i) \otimes \mathbb{Q}$$

is exact.

Proof. Clearly, the kernel of $i_{\#} \otimes 1: \pi_n(X_{\mathbb{Q}}) \rightarrow \pi_n(Y_{\mathbb{Q}})$ is the subspace $\mathbb{Q}(\alpha_{\mathbb{Q}})$ of $\pi_n(X_{\mathbb{Q}})$. If $\alpha_{\mathbb{Q}} = 0$, then $i_{\#} \otimes 1: \pi_n(X_{\mathbb{Q}}) \rightarrow \pi_n(Y_{\mathbb{Q}})$ is injective, and if $\alpha_{\mathbb{Q}} \notin G_n(X_{\mathbb{Q}})$, then the restriction of $i_{\#} \otimes 1$ to $G_n(X_{\mathbb{Q}})$ of $i_{\#}$ is injective. In either case, $H_n^{a\omega}(Y, X; i) \otimes \mathbb{Q} = 0$ by definition. Suppose next that (1) and (2) hold. Then n must be odd since $G_{2k}(X) \otimes \mathbb{Q} = 0$ for all k by [2, Th. III]. Suppose (3) does not hold, so that $Y \simeq_{\mathbb{Q}} *$. In this case we must have $X \simeq_{\mathbb{Q}} S^n$ for n odd with α a non-trivial class in $\pi_n(S^n) \otimes \mathbb{Q}$. Now it is straightforward to check that the map $H(P): G_{n+1}^{\text{rel}}(\mathcal{L}_Y, \mathcal{L}_X; \mathcal{L}_i) \rightarrow G_n(\mathcal{L}_X)$ is an isomorphism.

So suppose (1)–(3) hold; again, n is odd. Let $\text{Hur}_{\mathbb{Q}}: \pi_*(X) \rightarrow H_*(X; \mathbb{Q})$ denote the rational Hurewicz homomorphism and consider two cases. First suppose that $\text{Hur}_{\mathbb{Q}}(\alpha) = 0$. We show that $H(\text{ad}, \text{ad}_{\mathcal{L}_i}): H_n(\text{Rel}(\mathcal{L}_i)) \rightarrow H_n(\text{Rel}((\mathcal{L}_i)_*))$ is injective and conclude that $G_{n+1}^{\text{rel}}(Y, X; i) \otimes \mathbb{Q} = 0$ from Theorem 5.2. Write the Quillen model for X as $\mathbb{L}(W; d_X)$. A Quillen model for $i: X \rightarrow Y$ is an inclusion $\mathbb{L}(W) \rightarrow \mathbb{L}(W) \sqcup \mathbb{L}(y)$ with y of degree n . The differential d for $\mathbb{L}(W) \sqcup \mathbb{L}(y)$ satisfies $d(y) = \chi \in \mathbb{L}(W)$ a cycle of degree $(n-1)$ in \mathcal{L}_X whose homology class represents $\alpha_{\mathbb{Q}}$. Since $\text{Hur}_{\mathbb{Q}}(\alpha) = 0$, χ may be taken to be decomposable in \mathcal{L}_X . Thus $\mathbb{L}(W) \sqcup \mathbb{L}(y)$ with differential d is actually the Quillen minimal model for Y .

Any cycle $\zeta \in \text{Rel}_n(\mathcal{L}_i) = (\mathcal{L}_X)_{n-1} \oplus (\mathcal{L}_Y)_n$ may be written in the form $\zeta = (-d_Y(\lambda y + \xi), \lambda y + \xi)$ for $\lambda \in \mathbb{Q}$ and $\xi \in (\mathcal{L}_X)_n$. Suppose that $H(\text{ad}, \text{ad}_{\mathcal{L}_i})(\zeta) = 0$. Then $(\text{ad}, \text{ad}_{\mathcal{L}_i})(\zeta) = \delta(\theta, \varphi)$ for some $(\theta, \varphi) \in \text{Rel}_{n+1}((\mathcal{L}_i)_*)$. In particular, we have $\text{ad}_{\mathcal{L}_i}(\lambda y + \xi) = (\mathcal{L}_i)_*(\theta) + D\varphi \in \text{Der}_n(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_i)$. Now choose any indecomposable $w \in W$. On this indecomposable, we evaluate as follows:

$$\text{ad}_{\mathcal{L}_i}(\lambda y + \xi)(w) = \lambda[y, w] + [\xi, w] \quad (4)$$

and

$$((\mathcal{L}_i)_*(\theta) + D\varphi)(w) = \mathcal{L}_i \circ \theta(w) + d_Y \varphi(w) - (-1)^{n+1} \varphi d_X(w). \quad (5)$$

It is direct to check that all terms of (5) are independent of $[y, w]$. Thus the kernel of $H(\text{ad}, \text{ad}_{\mathcal{L}_i}): H_n(\text{Rel}(\mathcal{L}_i)) \rightarrow H_n(\text{Rel}((\mathcal{L}_i)_*))$ consists of classes represented by cycles of the form $\zeta = (-d_Y(\xi), \xi)$ for $\xi \in (\mathcal{L}_X)_n$. Since $d_Y(\xi) = d_X(\xi)$, we have $\delta(\xi, 0) = (-d_X(\xi), \xi) = \zeta$. So $H(\text{ad}, \text{ad}_{\mathcal{L}_i}): H_n(\text{Rel}(\mathcal{L}_i)) \rightarrow H_n(\text{Rel}((\mathcal{L}_i)_*))$ is injective and hence by Theorem 5.2, $G_{n+1}^{\text{rel}}(Y, X; i) \otimes \mathbb{Q} = 0$.

Finally, suppose that $\text{Hur}_{\mathbb{Q}}(\alpha) \neq 0$. In this case, a result due to Oprea ([10] but see also [5, Lem. 1.1]) implies we have a rational splitting $X \simeq_{\mathbb{Q}} X' \times S^n$ such that X' is non-trivial and α corresponds to a non-trivial class in $\pi_n(S^n) \otimes \mathbb{Q}$. From Corollary 2.2, we may write the Quillen model of X as $\mathbb{L}(W, v, W'; \partial)$, with v of degree $n-1$ and ∂ given by $\partial(v) = 0$, $\partial(w) = d_{X'}(w)$ and $\partial(w') = [v, w] - S(d_{X'}(w))$ (recall that n is odd). Then a Quillen model for $i: X \rightarrow Y$ is given by the projection $\mathcal{L}_i: \mathbb{L}(W, v, W') \rightarrow \mathbb{L}(W, W')$ defined by $\mathcal{L}_i(v) = 0$ and $\mathcal{L}_i(x) = x$ for $x \in W \oplus W'$. The differential ∂' in $\mathbb{L}(W, W')$ is the projection of ∂ , that is, $\partial' = \mathcal{L}_i \circ \partial$. If $\langle v \rangle \in G_n(\mathcal{L}_X)$ is in the image of $H(P): G_{n+1}^{\text{rel}}(\mathcal{L}_Y, \mathcal{L}_X; \mathcal{L}_i) \rightarrow G_n(\mathcal{L}_X)$, then without loss of generality we may assume that $(\text{ad}(v), 0) = \delta(\theta, \phi)$ for some $(\theta, \phi) \in \text{Rel}_{n+1}((\mathcal{L}_i)_*)$. We show that this leads to a contradiction. For this, pick an element $w_0 \in W$ of lowest degree, so that $\partial(w_0) = \partial'(w_0) = 0$, $\partial(w'_0) = [v, w_0]$, and $\partial'(w'_0) = 0$. The identity $D\theta = -\text{ad}(v)$ in $\text{Der}_{n-1}(\mathcal{L}_X, \mathcal{L}_X; 1)$ yields $\theta(w_0) = w'_0 + \alpha$ for some ∂ -cycle $\alpha \in \mathbb{L}(W, v)$. Then the identity $\mathcal{L}_i \circ \theta + D\phi = 0$ in $\text{Der}_n(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_i)$ implies that $w'_0 + \mathcal{L}_i(\alpha)$ is decomposable in $\mathbb{L}(W, W')$, which is a contradiction since $\mathcal{L}_i(\alpha) \in \mathbb{L}(W)$ and hence independent of the indecomposable $w'_0 \in W'$. Hence $H(P)(G_{n+1}^{\text{rel}}(\mathcal{L}_Y, \mathcal{L}_X; \mathcal{L}_i))$ does not contain $\langle v \rangle$ in this case either. \square

We conclude with an example of vanishing rational ω -homology. In the following result, we use the ideas discussed before Theorem 4.5, concerning the notion of a coformal map.

Theorem 5.5. *Let $f: X \rightarrow Y$ be a coformal map between CW complexes of finite type, with X finite. Then $H_n^{a\omega}(Y, X; f) \otimes \mathbb{Q} = 0$, that is,*

$$G_{n+1}^{\text{rel}}(Y, X; f) \otimes \mathbb{Q} \xrightarrow{p \otimes 1} G_n(X) \otimes \mathbb{Q} \xrightarrow{f_{\#} \otimes 1} G_n(Y, X; f) \otimes \mathbb{Q}$$

is exact, for each $n \geq 3$.

Proof. We will use Theorem 5.2 and show that $\ker\{H(\mathcal{L}_f)\} \subseteq \text{im}\{H(P)\}$. From Definition 4.4, we assume that both \mathcal{L}_X and \mathcal{L}_Y admit upper (second) gradings with the properties described in Definition 4.3, and that \mathcal{L}_f preserves upper degrees. Let $\alpha = \langle \xi \rangle \in G_{n+1}(\mathcal{L}_X)$ satisfy $H(\mathcal{L}_f)(\alpha) = 0$. We assume ξ is of upper degree zero in the bigraded model

for \mathcal{L}_X . Furthermore, $\mathcal{L}_f(\xi) = d_Y(y)$ for some $y \in (\mathcal{L}_Y)_{n+1}$ that we may assume is of upper degree 1. Since α is Gottlieb, $\text{ad}(\xi) = D(\psi)$ for some derivation $\psi \in \text{Der}_{n+1}(\mathcal{L}_X, \mathcal{L}_X; 1)$ and using the coformality of \mathcal{L}_X again, we may assume ψ increases upper degree homogeneously by 1. The pair $(\xi, -y) \in \text{Rel}_{n+1}(\mathcal{L}_f)$ is a δ -cycle that satisfies $P(\xi, -y) = \xi$. We now show that $(\xi, -y)$ represents an element in $G_{n+2}^{\text{rel}}(\mathcal{L}_Y, \mathcal{L}_X; \mathcal{L}_f)$, that is, we show the pair $(\text{ad}(\xi), -\text{ad}_{\mathcal{L}_f}(y))$ bounds in $\text{Rel}_{n+1}((\mathcal{L}_f)_*)$. Set $\Theta = -(\mathcal{L}_f)_*(\psi) + \text{ad}_{\mathcal{L}_f}(y)$, a derivation in $\text{Der}_{n+1}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$. It is direct to check that $D(\Theta) = d_Y \circ \Theta - (-1)^{n+1} \Theta \circ d_X = 0$. Moreover, Θ increases upper degree homogeneously by 1. Now adapt the proof of Theorem 4.5 to the current situation, by replacing the derivation $\text{ad}_{\mathcal{L}_f}(\xi)$ in that proof by Θ . The inductive argument used there now results in a derivation $\theta \in \text{Der}_{n+2}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$, constructed in the same way only increasing upper degree by 2, that satisfies $\Theta = D(\theta)$. Then the pair $(-\psi, -\theta) \in \text{Rel}_{n+2}((\mathcal{L}_f)_*)$ satisfies $\delta(-\psi, -\theta) = (\text{ad}(\xi), -\text{ad}_{\mathcal{L}_f}(y))$. \square

Appendix A. Some DG Lie algebra homotopy theory

The main results of this paper use a DG Lie algebra counterpart to a result of [8]; we now establish this result. Here, the main new ingredient is to show that the relation of homotopy under a map is preserved through the passage from Sullivan to Quillen models. Since this is a technical appendix, we assume a greater degree of familiarity with techniques from rational homotopy theory than in the main body of the paper. We use the notation and vocabulary from the appendix of [8] without comment.

Our main reference for this material is [1]; see also [13, Ch. II.5] and part IV of [3]. The algebraic notion of homotopy that we use here is defined in terms of a cylinder object. (The notion is discussed as “left homotopy” of DG Lie algebra maps in [13], although only in the absolute case.) In the category of DG Lie algebras, a *cofibration* corresponds to a map of DG Lie algebras of the form $L \rightarrow (L \sqcup \mathbb{L}(V), d)$, where L is a sub-DG Lie algebra of $(L \sqcup \mathbb{L}(V), d)$. Some authors refer to this as a *Koszul–Quillen extension*, or a *free extension*. We use an arrow of the form “ \rightrightarrows ” to indicate that the map is a cofibration. The results of [1], particularly in Sections II.1 and II.2 may be specialized to the context of DG Lie algebras using the dictionary “cofibration” \equiv free extension, “weak equivalence” \equiv quasi-isomorphism, “fibrant” \equiv any DG Lie algebra, and “cofibrant” \equiv free DG Lie algebra.

Given a cofibration $i: L \rightrightarrows M$ and a map $g: L \rightarrow N$, we say a map $G: M \rightarrow N$ is a *map under* g if $G \circ i = g$. The pushout

$$\begin{array}{ccc} L & \xrightarrow{i} & M \\ i \downarrow & & \downarrow i_2 \\ M & \xrightarrow{i_1} & M \sqcup_L M \end{array} \quad (6)$$

defines a folding map $\nabla: M \sqcup_L M \rightarrow M$. (See [1, I.9.12] for pushouts in the DG Lie algebra category.) Given maps $G, G': M \rightarrow N$ under g , we also obtain a map $(G \mid G'): M \sqcup_L M \rightarrow N$. A fact that we use frequently here is that any map of DG Lie algebras may be factored as a cofibration followed by a quasi-isomorphism (adapt [1, I.7.21], see also [3, (22.10)]). Given such a factorization of ∇ as

$$M \sqcup_L M \xrightarrow{c} I_L M \xrightarrow{\cong} M,$$

we say that c is a *cylinder object for* i . Then maps G and G' are *homotopic under* g if there is a *homotopy under* g , that is, a map $H: I_L M \rightarrow N$, such that $H \circ c = (G \mid G')$. Since every DG Lie algebra is “fibrant”, [1, II.2.2] implies that any convenient cylinder object may be used.

We describe the cylinder that we use in the proof of Theorem 3.1. Suppose X has Quillen minimal model $\mathbb{L}(W; d_X)$. Then $\mathbb{L}(W, v, W'; \partial)$, with generators and differential as specified in Corollary 2.2, is a model for $S^n \times X$. Evidently, the inclusion $i: S^n \vee X \rightarrow S^n \times X$ has Quillen minimal model $\mathcal{L}_i: \mathbb{L}(W, v) \rightarrow \mathbb{L}(W, v, W'; \partial)$ given by the obvious inclusion.

Example A.1. We follow [1, I.9.18] for the description of a cylinder for the cofibration $\mathcal{L}_i: \mathbb{L}(W, v) \rightarrow \mathbb{L}(W, v, W'; \partial)$. Form the pushout as in (6) to obtain

$$\mathbb{L}(W, v, W') \sqcup_{\mathbb{L}(W, v)} \mathbb{L}(W, v, W') = \mathbb{L}(W, v, W'_1, W'_2; \partial).$$

Here, W'_1 and W'_2 denote copies of W' and the differential ∂ extends that of $\mathbb{L}(W, v, W'; \partial)$ in the obvious way. Then the folding map ∇ factors as follows:

$$\begin{array}{ccc} \mathbb{L}(W, v, W'_1, W'_2; \partial) & \xrightarrow{\nabla} & \mathbb{L}(W, v, W'; \partial) \\ & \searrow c \quad \nearrow q \cong & \\ & \mathbb{L}(W, v, W', sW', \widehat{W}'; \partial_I) & \end{array}$$

Here, sW' is the suspension of W' and \widehat{W}' denotes a copy of W' . The differential ∂_I extends that of $\mathbb{L}(W, v, W'; \partial)$, we make the obvious modification of notation from Corollary 2.2 and write here $\partial_I(w') = (-1)^{n-1}[v, w] + (-1)^n S_I(\partial_I(w))$. On the other generators we have $\partial_I(\widehat{W}') = 0$ and $\partial_I(sw') = \widehat{w}'$ for each $w' \in W'$. Furthermore, we have $c(w) = w$, $c(v) = v$, and $c(w'_i) = w'$. To identify $c(w'_2)$, we introduce the following notation. Let

$$\sigma: \mathbb{L}(W, v, W', sW', \widehat{W}') \rightarrow \mathbb{L}(W, v, W', sW', \widehat{W}')$$

denote the derivation of degree +1 defined on generators by $\sigma(w') = sw'$ for each $w' \in W'$ and $\sigma = 0$ on all other generators. Then $[\partial_I, \sigma] = \partial_I \circ \sigma + \sigma \circ \partial_I$ is a derivation of degree 0 on $\mathbb{L}(W, v, W', sW', \widehat{W}')$ that may be exponentiated (it is “locally nilpotent”) to obtain an automorphism $e^{[\partial_I, \sigma]}$ of $\mathbb{L}(W, v, W', sW', \widehat{W}')$. To complete our description of the cylinder, we set $c(w'_2) = e^{[\partial_I, \sigma]}(w')$ for each $w'_2 \in W'_2$. The quasi-isomorphism q is the obvious projection with $q(sW') = 0$, $q(\widehat{W}') = 0$. Then c is our cylinder for the inclusion \mathcal{L}_i .

We may be more explicit about the cylinder described above. From the definition of the derivation $[\partial_I, \sigma]$, we find that $[\partial_I, \sigma] = 0$ on all generators other than those of W' , where we have $[\partial_I, \sigma](w') = \widehat{w}' + \sigma(\partial_I(w'))$. Furthermore, we have $\sigma \circ \sigma = 0$ – to confirm this identity, use $\sigma \circ \sigma = \frac{1}{2}[\sigma, \sigma]$, so that $\sigma \circ \sigma$ acts as a derivation – and $[\partial_I, \sigma](\widehat{w}') = 0$. It follows that the expression for $e^{[\partial_I, \sigma]}(w')$ reduces somewhat, to yield

$$c(w'_2) = w' + \widehat{w}' + \sum_{r \geq 1} \frac{1}{r} (\sigma \circ \partial_I)^r(w'). \quad (7)$$

This latter formula is actually valid in any cylinder (cf. the expression used in [13, II.5.1]). For our particular situation, there is a further simplification:

Lemma A.2. *In the cylinder described above, for a Quillen minimal model \mathcal{L}_i of the inclusion $i: S^n \vee X \rightarrow S^n \times X$, we have $(\sigma \circ \partial_I)^r = 0$ for $r \geq 2$ and hence*

$$c(w'_2) = w' + \widehat{w}' + \sigma(\partial_I(w')).$$

Proof. Because σ is zero on $\mathbb{L}(W, v)$, we have $\sigma(\partial_I(w')) = \sigma((-1)^{n-1}[v, w] + (-1)^n S_I(\partial_I(w))) = \pm \sigma \circ S_I(\partial_I(w)) \in \mathbb{L}(W, \widehat{W}')$. Since $\partial_I(\widehat{W}') = 0$ and $\mathbb{L}(W)$ is a sub-DG Lie algebra, $\mathbb{L}(W, \widehat{W}')$ is stable under ∂_I . Furthermore, $\sigma(W) = 0$ and $\sigma(\widehat{W}') = 0$. Hence $(\sigma \circ \partial_I)^2 = 0$. \square

For the proof of Theorem 3.1, we apply the following result to the inclusion $i: S^n \vee X \rightarrow S^n \times X$ and $g = (* \mid f): S^n \vee X \rightarrow Y$.

Proposition A.3. *Choose and fix a minimal model $\mathcal{L}_g: \mathcal{L}_U \rightarrow \mathcal{L}_Y$ for a map $g: U \rightarrow Y$. Let $i: U \rightarrow Z$ be a cofibration that induces an injection in rational homology groups. Then each map $F: Z \rightarrow Y$ under g has a minimal model $\mathcal{L}_F: \mathbb{L}(W, v, W'; \partial) \rightarrow \mathcal{L}_Y$ that is a map under \mathcal{L}_g . If F and F' are homotopic under g , then \mathcal{L}_F and $\mathcal{L}_{F'}$ are homotopic under \mathcal{L}_g .*

In the appendix to [8], we used the Sullivan functor, denoted $A^*(-)$, to pass from spaces to DG algebras. Here, we use the Quillen functor to pass from DG algebras to DG Lie algebras (see [3, Sec. 22(e)] or [13, I.1.(7)] for details). In fact, we only use general properties of this functor: it is contravariant; it preserves quasi-isomorphisms; it takes a surjective DG algebra map to a cofibration. In the following, we use \mathcal{L} to denote the composite of the Sullivan and the Quillen functors. That is, we write $\mathcal{L}(X)$ to denote the DG Lie algebra obtained by applying the Quillen functor to $A^*(X)$. Note that this departs from the convention of [3], for instance, in which \mathcal{L} is used to denote the Quillen functor itself.

Lemma A.4. Suppose given a cofibration $i: X \rightarrow Z$, a map $f: X \rightarrow Y$, and maps $F, G: Z \rightarrow Y$ homotopic under f . Then $\mathcal{L}(F), \mathcal{L}(G): \mathcal{L}(Z) \rightarrow \mathcal{L}(Y)$ are DG Lie homotopic as maps under $\mathcal{L}(f): \mathcal{L}(X) \rightarrow \mathcal{L}(Y)$.

Proof. Adjust the argument of [8, Lem. A.3], allowing for the fact that \mathcal{L} is covariant whereas $A^*(-)$ is contravariant. \square

It remains to show that the relation of homotopy under is preserved through the passage to minimal models in the DG Lie algebra setting. We begin by recalling the construction of the Quillen minimal model $\mathcal{L}_f: \mathcal{L}_X \rightarrow \mathcal{L}_Y$ of a map $f: X \rightarrow Y$. Let $\zeta_X: \mathcal{L}_X \rightarrow \mathcal{L}(X)$ and $\zeta_Y: \mathcal{L}_Y \rightarrow \mathcal{L}(Y)$ be Quillen minimal models of the spaces. Factor ζ_Y as a quasi-isomorphism followed by a surjection, to obtain

$$\mathcal{L}_Y \xleftarrow[\alpha_Y]{\beta_Y} \mathcal{L}_Y \sqcup E(\mathcal{L}(Y)) \xrightarrow{\gamma_Y} \mathcal{L}(Y). \quad (8)$$

The notation $E(L)$ for a DG Lie algebra L denotes the acyclic DG Lie algebra $\mathbb{L}(W, DW)$ with W isomorphic to the underlying module of L . The quasi-isomorphism α_Y is simply the inclusion of \mathcal{L}_Y as a DG Lie subalgebra, and the surjection γ_Y is the unique map that restricts to ζ_Y on \mathcal{L}_Y and the identity on the generators of $E(\mathcal{L}(Y))$. In fact, any map of DG Lie algebras may be factored in such a way. This construction is the DG Lie algebra analogue of the factorization described in [3, Sec. 12(b)] for DG algebras. It is described in [1, I.7.13] in the setting of DG chain algebras. Since ζ_Y itself is a quasi-isomorphism, so too is γ_Y . Furthermore β_Y , the obvious retraction of α_Y indicated, is also a surjective quasi-isomorphism. Since γ_Y is thus a surjective quasi-isomorphism, we may lift $\mathcal{L}(f) \circ \zeta_X$ through it to obtain a map $\phi_f: \mathcal{L}_X \rightarrow \mathcal{L}_Y \sqcup E(\mathcal{L}(Y))$ that satisfies $\gamma_Y \circ \phi_f = \mathcal{L}(f) \circ \zeta_X$. We set $\mathcal{L}_f = \beta_Y \circ \phi_f$.

Proof of Proposition A.3. Consider the diagrams

$$\begin{array}{ccc} \mathcal{L}_X & \xrightarrow[\simeq]{\zeta_X} & \mathcal{L}(X) \\ \mathcal{L}_i \downarrow & & \downarrow \mathcal{L}(i) \\ \mathcal{L}_Z & \xrightarrow[\zeta_Z]{\simeq} & \mathcal{L}(Z) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{L}_X & \xrightarrow[\simeq]{\zeta_X} & \mathcal{L}(X) \\ \phi \downarrow & & \downarrow \mathcal{L}(i) \\ \mathcal{L}_X \sqcup \mathbb{L}(V) & \xrightarrow[\simeq]{} & \mathcal{L}(Z). \end{array} \quad (9)$$

We begin by choosing a model for the cofibration $i: X \rightarrow Z$ such that the left-hand diagram commutes. The map $\mathcal{L}(i): \mathcal{L}(X) \rightarrow \mathcal{L}(Z)$ is a map of free Lie models and as such encodes the induced homomorphism $H_*(i; \mathbb{Q})$ as the (suspension of the) homomorphism induced on homology of the DG modules of indecomposables (cf. [3, (24.3)]). Then we factor $\mathcal{L}(i) \circ \zeta_X: \mathcal{L}_X \rightarrow \mathcal{L}(Z)$ as a cofibration followed by a quasi-isomorphism to obtain the right-hand commutative diagram. If this construction is carried out in a “minimal” way, as described in [3, Sec. 22(f)], for instance, then the condition of injectivity on $H_*(i; \mathbb{Q})$ results in the free extension $\mathcal{L}_X \sqcup \mathbb{L}(V)$ being minimal as a DG Lie algebra, that is, the differential is decomposable. So we may take this as the minimal model of Z .

Next we observe that a map $F: Z \rightarrow Y$ under $f: X \rightarrow Y$ has a Quillen minimal model \mathcal{L}_F that is a map under a given choice of \mathcal{L}_f . For suppose $\phi_f: \mathcal{L}_X \rightarrow \mathcal{L}_Y \sqcup E(\mathcal{L}(Y))$ is a given lift of $\mathcal{L}(f) \circ \zeta_X$ through γ_Y that defines a fixed choice of Quillen model of f as $\mathcal{L}_f = \beta_Y \circ \phi_f$, as above. Since the left-hand diagram of (9) commutes, we have the following commutative diagram of solid arrows

$$\begin{array}{ccc} \mathcal{L}_X & \xrightarrow{\phi_f} & \mathcal{L}_Y \sqcup E(\mathcal{L}(Y)) \\ \mathcal{L}_i \downarrow & \nearrow \phi_F & \downarrow \gamma_Y \\ \mathcal{L}_Z & \xrightarrow{\mathcal{L}(F) \circ \zeta_Z} & \mathcal{L}(Y). \end{array}$$

A lift ϕ_F as indicated exists, that is also a map under ϕ_f , by [3, Prop.2.11]. Such a lift is unique, up to DG Lie homotopy under ϕ_f , by [1, II.1.11(c)]. Then we have $\mathcal{L}_F \circ \mathcal{L}_i = \beta_Y \circ \phi_F \circ \mathcal{L}_i = \beta_Y \circ \phi_f = \mathcal{L}_f$, as claimed.

The remainder of the proof follows the same steps as the last part of the proof of [8, Lem. A.2], hence we omit it. \square

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